Published by Institute of Physics Publishing for SISSA/ISAS
Received: May 14, 2004
Accepted: May 19, 2004

On the relation between non-commutative field theories at $\theta = \infty$ and large N matrix field theories

Wolfgang Bietenholz and Frank Hofheinz

Institut für Physik, Humboldt Universität zu Berlin
Newtonstr. 15, D-12489 Berlin, Germany
E-mail: bietenho@physik.hu-berlin.de, hofheinz@physik.hu-berlin.de

Jun Nishimura

High Energy Accelerator Research Organization (KEK)
1-1 Oho, Tsukuba 305-0801, Japan
E-mail: jnishi@post.kek.jp

Abstract: It is well-known that non-commutative (NC) field theories at $\theta = \infty$ are "equivalent" to large N matrix field theories to all orders in perturbation theory, due to the dominance of planar diagrams. By formulating a NC field theory on the lattice non-perturbatively and mapping it onto a twisted reduced model, we point out that the above equivalence does not hold if the translational symmetry of the NC field theory is broken spontaneously. As an example we discuss NC scalar field theory, where such a spontaneous symmetry breakdown has been confirmed by Monte Carlo simulations.

Keywords: Spontaneous Symmetry Breaking, Non-Commutative Geometry, Matrix Models.

Contents

1. Introduction

Field theories on non-commutative (NC) spaces, which are characterized by the commutation relation among the coordinate operators $\hat{x}_\mu$ ($\mu = 1, \ldots, D$),

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\Theta_{\mu\nu}\,, \tag{1.1}$$

have recently attracted much attention in the context of string theory and quantum gravity. In particular perturbative aspects of such theories have been discussed extensively in the literature. Diagrammatically the difference from ordinary field theories in the commutative space is represented by a momentum dependent phase factor of the form

$$\exp\left(-\frac{i}{2}\,\Theta_{\mu\nu}\sum_{i<j} p_\mu^{(i)} p_\nu^{(j)}\right) \tag{1.2}$$

assigned to each vertex, where $\Theta_{\mu\nu}$ is the non-commutativity tensor in eq. (1.1), and $p_\mu^{(i)}$ ($i = 1, \ldots, n$) represent the momenta flowing into the vertex. The phase factors (except those depending only on the external momenta) cancel in the planar diagrams, whereas in the non-planar diagrams, at least one phase factor depending on an internal momentum remains.

Let θ be the typical magnitude of the non-zero elements of the non-commutativity tensor $\Theta_{\mu\nu}$. If we take the $\theta \to \infty$ limit (after removing the phase factor depending only on the external momenta), the non-planar diagrams vanish due to the oscillating phase, and we are left with the planar diagrams. In this way the $\theta \to \infty$ limit of NC field theories is "equivalent" to the large N matrix field theory in the *commutative* space to all orders in perturbation theory.

In this letter we point out that this well-known "equivalence" does not always hold *non-perturbatively*. In order to discuss this issue, we definitely need a non-perturbative formulation of NC field theory. As in ordinary field theories, we may regularize a NC field theory non-perturbatively by putting it on a lattice [1]. This is achieved by imposing the operator identity

$$e^{2\pi i \hat{x}_\mu / a} = \mathbb{1}\,, \tag{1.3}$$

where a is the lattice spacing, and the lattice sites are "fuzzy" due to the non-commutativity (1.1). Using a correspondence between a field on a finite lattice and a matrix, which is *exact* in the sense that the *finite* degrees of freedom on one side are mapped onto the other, one to one, we can map a lattice NC field theory onto the twisted reduced model [1]. This is a refinement of the earlier work [2], where continuum NC theory is mapped to the continuum version of the twisted reduced model [3], which is formally defined in terms of infinite-dimensional matrices.

Reduced models appeared in history first as an equivalent description of large N gauge theory in the early 80s [4]. The original Eguchi-Kawai (EK) model is proved to be "equivalent" to SU(∞) lattice gauge theory under the assumption that the $\mathrm{U}(1)^D$ symmetry of the model is *not* spontaneously broken. It was found, however, that this symmetry *is* actually broken in the weak coupling regime at $D \geq 3$ [5]. The *twisted* EK model is one of the proposals to overcome this problem [6],[1] and the equivalence was confirmed by the weak coupling expansion as well as the strong coupling expansion. Soon later it was noticed that this idea of "twisted reduction" can be applied also to non-gauge theories written in terms of matrix fields [8, 9]. As in the case of gauge theory [4, 6], the proof based on the Schwinger-Dyson (SD) equations reveals that there are certain symmetries which should not be spontaneously broken in order for the equivalence to hold [9].

In the new interpretation of the twisted reduced model as a lattice regularization of NC field theories, the non-commutativity parameter θ is identified as [1]

$$\theta \propto N^{2/D} a^2\,, \tag{1.4}$$

where N is the size of the matrices that appear in twisted reduced models. In the context of EK equivalence, one has to take the large N limit at a fixed lattice spacing a (hence at fixed bare parameters with appropriate normalization with respect to N),[2] which corresponds to the $\theta = \infty$ limit of NC field theories as is clear from eq. (1.4). Thus the EK equivalence provides a non-perturbative account for the aforementioned "equivalence" between the NC field theories at $\theta = \infty$ and the large N matrix field theories. We will see that the condition for this "equivalence" is — in the language of NC field theories — that the translational symmetry is not spontaneously broken.

[1] For other proposals, which may be suitable for numerical studies of planar QCD, see refs. [7].

[2] This large N limit is referred to as the *planar* large N limit, since in perturbation theory only planar diagrams survive this limit. On the other hand, in order to obtain the continuum limit of NC field theories with finite θ, one has to take the large N limit and the $a \to 0$ limit simultaneously, which is usually referred to as the *double scaling* limit. Indeed various correlation functions scale in this limit for simple models as demonstrated by Monte Carlo simulations [10]–[12].

JHEP05(2004)047

In fact in NC scalar field theories the spontaneous breakdown of translational invariance was conjectured in refs. [13, 14], and confirmed recently by Monte Carlo simulations in refs. [11, 15, 12]. Before we discuss the NC scalar field theory, we start in the next Section with a brief comment on the gauge theory case. For a comprehensive review on the EK equivalence we refer the reader to ref. [16], and for reviews on the novel connection between reduced models and NC field theories to refs. [17, 18].

2. The case of gauge theory

The twisted EK model [6] is defined by

$$S = -N\beta \sum_{1\leq\mu<\nu\leq D} Z_{\mu\nu}\ \mathrm{Tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) + \mathrm{c.c.}\,, \tag{2.1}$$

where U_μ ($\mu = 1, \ldots, D$) are $N \times N$ unitary matrices, and $Z_{\mu\nu} = (Z_{\nu\mu})^*$ is a phase factor which is referred to as the "twist". Here and henceforth we assume D to be an even integer. The model has the $\mathrm{SU}(N)$ symmetry

$$U_\mu \mapsto g\, U_\mu\, g^\dagger \tag{2.2}$$

and the $\mathrm{U}(1)^D$ symmetry

$$U_\mu \mapsto \mathrm{e}^{i\alpha_\mu} U_\mu\,. \tag{2.3}$$

As $\mathrm{SU}(N)$ covariant quantities we define

$$w(\mu, \nu, \ldots, \sigma) = \frac{1}{N} U_\mu U_\nu \cdots U_\sigma\,, \tag{2.4}$$

where the Greek indices may take the values $\pm 1 \ldots \pm D$, and we define $U_{-\mu} = U_\mu^\dagger$. For each term $w(\mu, \nu, \ldots, \sigma)$, we introduce the displacement vector

$$\vec{v} = \hat{\mu} + \hat{\nu} + \cdots + \hat{\sigma}\,, \tag{2.5}$$

where $\hat{\mu}$ is a unit vector in the μ direction. Under the $\mathrm{U}(1)^D$ transformation (2.3) the quantity (2.4) transforms as

$$w(\mu, \nu, \ldots, \sigma) \mapsto \mathrm{e}^{i\vec{v}\cdot\vec{\alpha}}\, w(\mu, \nu, \ldots, \sigma)\,, \tag{2.6}$$

and $\mathrm{Tr}\, w(\mu, \nu, \cdots, \sigma)$ is $\mathrm{SU}(N)$ invariant. The EK equivalence states that in the planar large N limit the vacuum expectation value (VEV)

$$W(\mu, \nu, \ldots, \sigma) = \left(\prod_{\lambda\rho} (Z_{\lambda\rho})^{P_{\lambda\rho}}\right) \langle \mathrm{Tr}\, w(\mu, \nu, \ldots, \sigma)\rangle \tag{2.7}$$

for $\vec{v} = 0$ agrees with the corresponding Wilson loop in the $\mathrm{SU}(\infty)$ lattice gauge theory, where $P_{\lambda\rho}$ denotes the number of plaquettes in the (λ, ρ) plane within the surface which spans the Wilson loop. The equivalence requires all the VEVs $\langle \mathrm{Tr}\, w(\mu, \nu, \ldots, \sigma)\rangle$ for $\vec{v} \neq 0$ to vanish, which is satisfied if the $\mathrm{U}(1)^D$ symmetry (2.3) is not spontaneously broken.

JHEP05(2004)047

Now we interpret the twisted reduced model (2.1) as a NC gauge theory [1]. Note that the configuration which minimizes the action (2.1) is given by $U_\mu = \Gamma_\mu$, where Γ_μ are $\mathrm{SU}(N)$ matrices satisfying the 't Hooft-Weyl algebra[3]

$$\Gamma_\mu \Gamma_\nu = Z_{\nu\mu} \Gamma_\nu \Gamma_\mu \, . \tag{2.8}$$

This motivates us to make a substitution

$$U_\mu = \tilde{U}_\mu \Gamma_\mu \, , \tag{2.9}$$

which brings the action (2.1) into the form

$$S = -N\beta \sum_{1 \le \mu < \nu \le D} \mathrm{Tr} \left[\tilde{U}_\mu (\Gamma_\mu \tilde{U}_\nu \Gamma_\mu^\dagger)(\Gamma_\nu \tilde{U}_\mu \Gamma_\nu^\dagger)^\dagger \tilde{U}_\nu^\dagger \right] + \text{c.c.} \, . \tag{2.10}$$

Then we consider a one-to-one correspondence from the $N \times N$ matrices $\tilde{U}_\mu$ to the lattice gauge field of rank 1 on a L^D lattice, where $N^2 = L^D$ due to the matching of the degrees of freedom. In this correspondence, the matrix product becomes the so-called "star-product" of fields on the lattice, and the transformation

$$\tilde{U}_\mu \mapsto \Gamma_\nu \tilde{U}_\mu \Gamma_\nu^\dagger \tag{2.11}$$

represents a shift of the lattice gauge field in the ν direction by one lattice unit. Thus the expression (2.10) can be regarded as the "plaquette action" for the NC gauge theory on the lattice. Note also that the $\mathrm{SU}(N)$ symmetry (2.2) of the twisted reduced model can be re-written as

$$\tilde{U}_\mu \mapsto g \, \tilde{U}_\mu \, (\Gamma_\mu \, g \, \Gamma_\mu^\dagger)^\dagger \, , \tag{2.12}$$

which represents the "gauge symmetry" of the lattice NC gauge theory.

The action (2.10) is invariant under the transformation (2.11), which represents the translational invariance of the NC gauge theory. The whole symmetry group generated by such transformations is $(\mathrm{Z}_L)^D$, whose elements are given explicitly as

$$\begin{aligned} \tilde{U}_\mu &\mapsto \Gamma(\vec{x}) \, \tilde{U}_\mu \, \Gamma^\dagger(\vec{x}) & (2.13) \\ \Gamma(\vec{x}) &= (\Gamma_1)^{x_1} (\Gamma_2)^{x_2} \cdots (\Gamma_D)^{x_D} \, . & (2.14) \end{aligned}$$

Note here that $(\Gamma_\mu)^L = \mathbb{1}$, so the L^D lattice is periodic. In terms of original variables in the twisted EK model, the transformation (2.11) reads

$$U_\mu \mapsto Z_{\nu\mu} \, \Gamma_\nu \, U_\mu \, \Gamma_\nu^\dagger \, , \tag{2.15}$$

which can be realized by combining a $\mathrm{SU}(N)$ transformation (2.2) and a $\mathrm{U}(1)^D$ transformation (2.3). In other words, the $\mathrm{U}(1)^D$ symmetry of the twisted EK model includes the translational symmetry of the NC gauge theory up to the "gauge symmetry" (2.12). Thus

[3]Explicit forms of the twist $Z_{\mu\nu}$ and the corresponding solution Γ_μ are given, for instance, in refs. [1].

in the new interpretation of the twisted EK model, the NC gauge theory at $\theta = \infty$ is equivalent to SU(∞) lattice gauge theory, *if* the translational invariance is not spontaneously broken.

It is known that the $\mathrm{U}(1)^D$ symmetry of the twisted EK model is unbroken in both, the weak coupling (large β) and the strong coupling (small β) regime [6]. However, there is a certain indication in $D = 4$ that the symmetry is broken at intermediate couplings [19], which might be related to the perturbative instabilities pointed out in ref. [20]. These issues should be clarified further.

3. The case of scalar field theory

Let us move on to the case of scalar field theory. For concreteness we discuss the $(D+1)$-dimensional NC scalar field theory, where non-commutativity is introduced only in D directions (D even), but it is straightforward to generalize our argument to the case with an arbitrary number of commutative directions (including the case with *no* commutative direction). The corresponding twisted reduced model can be given as [11, 12]

$$S_{\mathrm{TRM}} = N\,\mathrm{Tr}\sum_{t=1}^{T}\left[\frac{1}{2}\sum_{\mu=1}^{D}\Big(\Gamma_\mu\hat{\phi}(t)\Gamma_\mu^\dagger - \hat{\phi}(t)\Big)^2 + \frac{1}{2}\Big(\hat{\phi}(t+1)-\hat{\phi}(t)\Big)^2 + \frac{m^2}{2}\hat{\phi}(t)^2 + \frac{\lambda}{4}\hat{\phi}(t)^4\right], \tag{3.1}$$

where $\hat{\phi}(t)$ represents a hermitean $N \times N$ matrix living on one of the discrete time points $t = 1, \ldots, T$. The coupling constant λ is assumed to be real positive, while the mass squared m^2 can take any real number. The action (3.1) has the $(\mathrm{Z}_L)^D$ symmetry

$$\hat{\phi}(t) \mapsto \Gamma(\vec{x})\,\hat{\phi}(t)\,\Gamma^\dagger(\vec{x}), \tag{3.2}$$

which plays a crucial role.

The EK equivalence states that, in the large N limit with fixed m^2 and λ, the twisted reduced model (3.1) is "equivalent" to the matrix field theory

$$\begin{aligned} S_{\mathrm{MFT}}[\Phi] = N\,\mathrm{Tr}\sum_{t=1}^{T}\sum_{\vec{x}}\Bigg[&\frac{1}{2}\sum_{\mu=1}^{D}(\Phi(\vec{x}+\hat{\mu},t) - \Phi(\vec{x},t))^2 + \frac{1}{2}(\Phi(\vec{x},t+1) - \Phi(\vec{x},t))^2 + \\ &+ \frac{m^2}{2}\Phi(\vec{x},t)^2 + \frac{\lambda}{4}\Phi(\vec{x},t)^4\Bigg] \end{aligned} \tag{3.3}$$

on the $(D+1)$-dimensional hypercubic lattice, where $\Phi(\vec{x},t)$ represents a hermitean $N \times N$ matrix living on each site. This matrix field theory has the global SU(N) symmetry

$$\Phi(\vec{x},t) \mapsto g\,\Phi(\vec{x},t)\,g^\dagger, \tag{3.4}$$

where $g \in \mathrm{SU}(N)$. Let us consider a matrix field product

$$w(\vec{x}_1,t_1;\ldots;\vec{x}_k,t_k) = \frac{1}{N}\,\Phi(\vec{x}_1,t_1)\,\Phi(\vec{x}_2,t_2)\cdots\Phi(\vec{x}_k,t_k), \tag{3.5}$$

whose trace is invariant under a global SU(N) transformation (3.4).

In the twisted reduced model (3.1), we may define the corresponding quantity as

$$\tilde{w}(\vec{x}_1, t_1; \ldots; \vec{x}_k, t_k) = \frac{1}{N}\left(\Gamma(\vec{x}_1)\hat{\phi}(t_1)\Gamma^\dagger(\vec{x}_1)\right)\cdots\left(\Gamma(\vec{x}_k)\hat{\phi}(t_k)\Gamma^\dagger(\vec{x}_k)\right). \tag{3.6}$$

Let us denote the VEV with respect to the twisted reduced model (3.1) and the matrix field theory (3.3) by $\langle\cdots\rangle_{\mathrm{TRM}}$ and $\langle\cdots\rangle_{\mathrm{MFT}}$, respectively. Then the statement is that

$$\langle \mathrm{Tr}\,\tilde{w}(\vec{x}_1, t_1; \ldots; \vec{x}_k, t_k)\rangle_{\mathrm{TRM}} = \langle \mathrm{Tr}\,w(\vec{x}_1, t_1; \ldots; \vec{x}_k, t_k)\rangle_{\mathrm{MFT}} \tag{3.7}$$

in the large N limit with fixed m^2 and λ, if the $(\mathrm{Z}_L)^D$ symmetry (3.2) is not spontaneously broken. A proof based on the SD equations is presented in the appendix.

When we interpret the twisted reduced model (3.1) as a $(D+1)$-dimensional NC scalar field theory, we map the matrix configuration $\{\hat{\phi}(t)\}$ to a one-component real scalar field $\{\phi(\vec{x}, t)\}$ on the $L^D \times T$ lattice, where $N^2 = L^D$. In this correspondence, the $(\mathrm{Z}_L)^D$ symmetry (3.2) is nothing but the translational symmetry in the NC directions. Hence we conclude that the NC scalar field theory at $\theta = \infty$ is equivalent to the large N matrix field theory, provided that the translational symmetry in the NC directions is not spontaneously broken.

The $(2+1)$-dimensional case ($D = 2$) has been studied by Monte Carlo simulation in refs. [11, 12]. In the planar large N limit with m^2 below some (negative) critical value $m^2_{\mathrm{c,TRM}}(\lambda)$, the scalar field $\phi(\vec{x}, t)$ acquires a VEV,[4] which depends on the coordinates $\vec{x}$ in the NC directions. In this "striped phase", the translational symmetry is spontaneously broken,[5] so the equivalence to the large N matrix field theory is no longer valid.

On the other hand, in the planar large N limit of the matrix field theory (3.3) one expects a standard Ising-type phase transition at some critical point $m^2 = m^2_{\mathrm{c,MFT}}(\lambda)$, below which the matrix field $\Phi(\vec{x}, t)$ acquires a uniform VEV proportional to the unit matrix. From the simulation results of the twisted reduced model, we may conclude that $m^2_{\mathrm{c,TRM}}(\lambda) \geq m^2_{\mathrm{c,MFT}}(\lambda)$. If we assume the opposite, i.e. $m^2_{\mathrm{c,TRM}}(\lambda) < m^2_{\mathrm{c,MFT}}(\lambda)$, we should have observed a uniformly ordered phase for $m^2_{\mathrm{c,TRM}}(\lambda) < m^2 < m^2_{\mathrm{c,MFT}}(\lambda)$, which is not the case. *A priori* there is no compelling reason to believe that $m^2_{\mathrm{c,TRM}}(\lambda) = m^2_{\mathrm{c,MFT}}(\lambda)$.

In ref. [12] we have determined the effective mass M_{eff} (or the renormalized mass) squared from the dispersion relation $E^2 = M^2_{\mathrm{eff}} + \vec{p}^{\,2}$ in the planar large N limit of the twisted reduced model (3.1). We have seen that it behaves linearly with respect to the bare mass squared as

$$M^2_{\mathrm{eff}} \propto m^2 - m^2_{\mathrm{c,TRM}}(\lambda) \tag{3.8}$$

[4] As is always the case with spontaneous breakdown of symmetries, the VEV should be defined carefully. We introduce an explicit symmetry breaking term in the action, calculate the VEV in the thermodynamic limit, and finally remove the symmetry breaking term. It is assumed that the VEVs in this article are defined in this way whenever it is necessary.

[5] There is a numerical evidence for broken translational invariance in 2d NC scalar field theory (with no commutative "time" direction) as well [15] [12]. As ref. [15] pointed out, this does not contradict the Mermin-Wagner Theorem since the proof for that Theorem assumes properties like locality and a regular IR behavior, which do not hold in most NC field theories. The EK equivalence has been re-examined also in 2d chiral models [21].

JHEP05(2004)047

to a good accuracy as we approach the critical point $m^2_{\mathrm{c,TRM}}(\lambda)$ from above. Since the correlation length may be identified as $\xi = (M_{\mathrm{eff}})^{-1}$, our result (3.8) implies that the (standard) critical exponent ν, defined by

$$\xi \propto \left\{m^2 - m^2_{\mathrm{c,TRM}}(\lambda)\right\}^{-\nu} , \tag{3.9}$$

agrees with the mean field value $\nu = 1/2$. (More precisely, our data imply $\nu = 0.48(4)$.)

On the other hand, the critical exponent for the Ising-like phase transition in the 3d matrix field theory (3.3) is expected to deviate from the mean field value generically. For instance, refs. [22, 23] obtain $\nu \simeq 0.67$ from a wilsonian renormalization group study with a number of drastic simplifications. If we rely on that result, then the fact that we obtain the mean field value $\nu \simeq 1/2$ suggests that the phase transition in the NC scalar field theory at $\theta = \infty$ has nothing to do with the phase transition in the large N matrix field theory.

4. Summary and discussions

In this letter we pointed out that the "equivalence" between non-commutative field theories at $\theta = \infty$ and large N matrix field theories, which is commonly believed based on perturbation theory, does not hold when the translational symmetry is spontaneously broken. Our argument is based on the new interpretation of the twisted reduced models as the lattice NC field theories. In particular the equivalence does not hold in the striped phase of the NC scalar field theory.[6] Even in such a parameter region, if one formally expands the two theories around the classical vacuum with a *uniform* order, one would obtain the same perturbative series. What occurs here is that the NC theory actually chooses a different vacuum.

In the traditional context of EK equivalence, the spontaneous symmetry breaking (SSB) is problematic if it occurs at the points in the parameter space where the continuum limit should be taken. For instance, in the case of scalar field theory, one cannot study the continuum limit of the large N matrix field theories by using the twisted reduced models. In the case of gauge theory, the SSB might occur at some intermediate coupling, but this is harmless since the continuum limit is taken by extrapolating the bare coupling constant to zero along with $a \to 0$.

On the other hand, from the view point of NC field theories, the SSB of translational symmetry provides a possibility to construct NC field theories with finite θ which do not reduce to the corresponding large N matrix field theories in the $\theta \to \infty$ limit. The 3d NC scalar field theory studied in ref. [12] provides the first explicit example of this sort. This implies that the universality class of NC field theories can be different from that of large N matrix field theories.

[6]This does not agree with a statement in ref. [13] that the equivalence holds beyond the critical point, which is based on analytic continuation in the complex parameter space.

JHEP05(2004)047

Acknowledgments

It is our pleasure to thank Tomomi Ishikawa, Satoshi Iso, Hikaru Kawai, Yoshihisa Kitazawa and Yoshiaki Susaki for useful discussions. The work of J.N. was supported in part by Grant-in-Aid for Scientific Research (No. 14740163) from the Ministry of Education, Culture, Sports, Science and Technology.

A. A proof of the EK equivalence for scalar field theory

In this appendix we show that the EK equivalence (3.7) between the twisted reduced model (3.1) and the matrix field theory (3.3) holds in the large N limit with fixed m^2 and λ, if the $(\mathrm{Z}_L)^D$ symmetry (3.2) is not spontaneously broken. For that purpose, we consider the SD equations, which form a closed set of equations for the VEVs in eq. (3.7). The proof is analogous to the ones for other models [4, 6, 9].

In the matrix field theory (3.3), let us consider the quantity

$$\langle \mathrm{Tr}\{\lambda_a\, \Phi(\vec{x}_1,t_1)\, \Phi(\vec{x}_2,t_2)\cdots \Phi(\vec{x}_k,t_k)\}\rangle_{\mathrm{MFT}}\ , \tag{A.1}$$

where λ_a is a generator of $\mathrm{U}(N)$, and apply the change of variable

$$\Phi(\vec{x}_1,t_1)\mapsto \Phi(\vec{x}_1,t_1)+\epsilon\,\lambda_a\,, \tag{A.2}$$

where ϵ is an infinitesimal parameter. Summing over the index $a=1,\ldots,N^2$ and using the identity

$$\sum_{a=1}^{N^2}(\lambda_a)_{ij}\,(\lambda_a)_{kl}=\delta_{il}\,\delta_{jk} \tag{A.3}$$

and the large N factorization property, we obtain

$$\begin{aligned}
0 = &\Big\langle \mathrm{Tr}\, w(\vec{x}_2,t_2;\ldots;\vec{x}_k,t_k)\Big\rangle_{\mathrm{MFT}} + \\
&+\Big\langle \mathrm{Tr}\, w(\vec{x}_1,t_1;\ldots;\vec{x}_k,t_k)\times \\
&\quad\times\Big[\sum_{\mu=1}^{D}\{\Phi(\vec{x}_1+\hat{\mu},t_1)+\Phi(\vec{x}_1-\hat{\mu},t_1)-2\Phi(\vec{x}_1,t_1)\}+ \\
&\qquad+\{\Phi(\vec{x}_1,t_1+1)+\Phi(\vec{x}_1,t_1-1)-2\Phi(\vec{x}_1,t_1)\}- \\
&\qquad-m^2\Phi(\vec{x}_1,t_1)-\lambda\Phi(\vec{x}_1,t_1)^3\Big]\Big\rangle_{\mathrm{MFT}} + \\
&+\sum_{s=2}^{k}\delta_{\vec{x}_1,\vec{x}_s}\delta_{t_1 t_s}\Big\langle \mathrm{Tr}\, w(\vec{x}_1,t_1;\ldots;\vec{x}_{s-1},t_{s-1})\Big\rangle_{\mathrm{MFT}}\times \\
&\times\Big\langle \mathrm{Tr}\, w(\vec{x}_{s+1},t_{s+1};\ldots;\vec{x}_k,t_k)\Big\rangle_{\mathrm{MFT}}\,.
\end{aligned} \tag{A.4}$$

SD equations for the twisted reduced model (3.1) can be derived in a similar manner. Let us consider the quantity

$$\Big\langle \mathrm{Tr}\{\Gamma(\vec{x}_1)\lambda_a\hat{\phi}(t_1)\Gamma^\dagger(\vec{x}_1)\cdots\Gamma(\vec{x}_k)\hat{\phi}(t_k)\Gamma^\dagger(\vec{x}_k)\}\Big\rangle_{\mathrm{TRM}}, \tag{A.5}$$

and apply the change of variable

$$\hat{\phi}(t_1) \mapsto \hat{\phi}(t_1) + \epsilon\,\lambda_a\,. \tag{A.6}$$

Summing over the indices and using the large N factorization property, we obtain

$$\begin{aligned}
0 = &\Big\langle \mathrm{Tr}\,\tilde{w}(\vec{x}_2, t_2; \dots; \vec{x}_k, t_k)\Big\rangle_{\mathrm{TRM}} + \\
&+ \Big\langle \mathrm{Tr}\,\tilde{w}(\vec{x}_1, t_1; \dots; \vec{x}_k, t_k) \times \\
&\quad \times \Big[\sum_{\mu=1}^{D} \{\Gamma_\mu \hat{\phi}(t_1)\Gamma_\mu^\dagger + \Gamma_\mu^\dagger \hat{\phi}(t_1)\Gamma_\mu - 2\hat{\phi}(t_1)\} + \\
&\qquad + \{\hat{\phi}(t_1+1) + \hat{\phi}(t_1-1) - 2\hat{\phi}(t_1)\} - \\
&\qquad - m^2\hat{\phi}(t_1) - \lambda\hat{\phi}(t_1)^3\Big]\Big\rangle_{\mathrm{TRM}} + \\
&+ \sum_{s=2}^{k} \delta_{t_1,t_s} \Big\langle \mathrm{Tr}\,w(\vec{x}_1, t_1; \dots; \vec{x}_{s-1}, t_{s-1})\Gamma(\vec{x}_s)\Gamma^\dagger(\vec{x}_1)\Big\rangle_{\mathrm{TRM}} \times \\
&\times \Big\langle \mathrm{Tr}\,w(\vec{x}_{s+1}, t_{s+1}; \dots; \vec{x}_k, t_k)\Gamma(\vec{x}_1)\Gamma^\dagger(\vec{x}_s)\Big\rangle_{\mathrm{TRM}}\,.
\end{aligned} \tag{A.7}$$

Comparing eqs. (A.4) and (A.7), we find that the SD equations for the twisted reduced model are identical to those for the matrix field theory if and only if

$$\Big\langle \mathrm{Tr}\,\tilde{w}(\vec{x}_1, t_1; \cdots; \vec{x}_n, t_n)\Gamma(\vec{x})\Gamma^\dagger(\vec{x}_1)\Big\rangle_{\mathrm{TRM}} = 0 \qquad \text{for } \forall \vec{x} \neq \vec{x}_1\,. \tag{A.8}$$

Note that the operator on the left-hand-side transforms non-trivially under the $(\mathrm{Z}_L)^D$ transformation (3.2), which is a symmetry of the twisted reduced model. Therefore the SD equations coincide in the large N limit with fixed m^2 and λ, if the $(\mathrm{Z}_L)^D$ symmetry is not spontaneously broken. Assuming that the SD equations have a unique solution,[7] we obtain eq. (3.7).

References

[1] J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, *Finite N matrix models of noncommutative gauge theory*, *J. High Energy Phys.* **11** (1999) 029 [hep-th/9911041]; *Nonperturbative dynamics of noncommutative gauge theory*, *Phys. Lett.* **B 480** (2000) 399 [hep-th/0002158]; *Lattice gauge fields and discrete noncommutative Yang-Mills theory*, *J. High Energy Phys.* **05** (2000) 023 [hep-th/0004147].

[2] H. Aoki et al., *Noncommutative Yang-Mills in IIB matrix model*, *Nucl. Phys.* **B 565** (2000) 176 [hep-th/9908141].

[3] A. González-Arroyo and C.P. Korthals Altes, *Reduced model for large-N continuum field theories*, *Phys. Lett.* **B 131** (1983) 396.

[4] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063.

[7] Although this assumption is valid generically, there are special cases in which the solution is not unique [24].

[5] G. Bhanot, U.M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett.* **B 113** (1982) 47.

[6] A. González-Arroyo and M. Okawa, *The twisted Eguchi-Kawai model: a reduced model for large-N lattice gauge theory*, *Phys. Rev.* **D 27** (1983) 2397.

[7] J. Kiskis, R. Narayanan and H. Neuberger, *Proposal for the numerical solution of planar QCD*, *Phys. Rev.* **D 66** (2002) 025019 [hep-lat/0203005];
R. Narayanan and H. Neuberger, *Large-N reduction in continuum*, *Phys. Rev. Lett.* **91** (2003) 081601 [hep-lat/0303023].

[8] T. Eguchi and R. Nakayama, *Simplification of quenching procedure for large-N spin models*, *Phys. Lett.* **B 122** (1983) 59.

[9] A. González-Arroyo and M. Okawa, *Twisted reduced chiral model*, *Nucl. Phys.* **B 247** (1984) 104.

[10] W. Bietenholz, F. Hofheinz and J. Nishimura, *A non-perturbative study of gauge theory on a non-commutative plane*, *J. High Energy Phys.* **09** (2002) 009 [hep-th/0203151].

[11] W. Bietenholz, F. Hofheinz and J. Nishimura, *Simulating non-commutative field theory*, *Nucl. Phys.* **119** *(Proc. Suppl.)* (2003) 941 [hep-lat/0209021]; *Non-commutative field theories beyond perturbation theory*, *Fortschr. Phys.* **51** (2003) 745 [hep-th/0212258]; *The non-commutative $\lambda\phi^4$ model*, *Acta Phys. Polon.* **B34** (2003) 4711 [hep-th/0309216];
F. Hofheinz, *Field theory on a non-commutative plane: a non-perturbative study*, *Fortschr. Phys.* **52** (2004) 391 [hep-th/0403117].

[12] W. Bietenholz, F. Hofheinz and J. Nishimura, *Phase diagram and dispersion relation of the non-commutative $\lambda\phi^4$ model in $D = 3$*, hep-th/0404020.

[13] S.S. Gubser and S.L. Sondhi, *Phase structure of non-commutative scalar field theories*, *Nucl. Phys.* **B 605** (2001) 395 [hep-th/0006119].

[14] P. Castorina and D. Zappalà, *Nonuniform symmetry breaking in noncommutative $\lambda\phi^4$ theory*, *Phys. Rev.* **D 68** (2003) 065008 [hep-th/0303030].

[15] J. Ambjørn and S. Catterall, *Stripes from (noncommutative) stars*, *Phys. Lett.* **B 549** (2002) 253 [hep-lat/0209106].

[16] S.R. Das, *Some aspects of large-N theories*, *Rev. Mod. Phys.* **59** (1987) 235.

[17] R.J. Szabo, *Quantum field theory on noncommutative spaces*, *Phys. Rept.* **378** (2003) 207 [hep-th/0109162].

[18] J. Nishimura, *Lattice superstring and noncommutative geometry*, *Nucl. Phys.* **129** *(Proc. Suppl.)* (2004) 121 [hep-lat/0310019].

[19] T. Ishikawa and M. Okawa, *Z_N^D symmetry breaking on the numerical simulation of twisted Eguchi-Kawai model*, talk given by T. Ishikawa at the annual meeting of the Physical Society of Japan, March 2003.

[20] Z. Guralnik, R.C. Helling, K. Landsteiner and E. Lopez, *Perturbative instabilities on the non-commutative torus, Morita duality and twisted boundary conditions*, *J. High Energy Phys.* **05** (2002) 025 [hep-th/0204037].

[21] S. Profumo and E. Vicari, *Twisted Eguchi-Kawai reduced chiral models*, *J. High Energy Phys.* **05** (2002) 014 [hep-th/0203155].

JHEP05(2004)047

[22] G. Ferretti, *On the large-N limit of 3D and 4D hermitean matrix models*, *Nucl. Phys.* **B 450** (1995) 713 [hep-th/9504013].

[23] S. Nishigaki, *Wilsonian approximated renormalization group for matrix and vector models in* $2 < D < 4$, *Phys. Lett.* **B 376** (1996) 73 [hep-th/9601043].

[24] J. Nishimura, *On existence of nontrivial fixed points in large-N gauge theory in more than four dimensions*, *Mod. Phys. Lett.* **A 11** (1996) 3049 [hep-lat/9608119].